

Question 5

$$L(\cos x) = L\left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}\right)$$

$$L(e^x) = L\left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)$$

Taking Laplace Transform term by term gives

$$L(\cos x) = \int_0^{\infty} e^{-sx} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} dx$$

$$L(e^x) = \int_0^{\infty} e^{-sx} \sum_{n=0}^{\infty} \frac{x^n}{n!} dx$$

$$\text{Then } L(\cos x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{(2n+1)}}{(2n+1)!} \int_0^{\infty} e^{-sx} dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{(2n+1)}}{(2n+1)!} \cdot \frac{1}{s}$$

$$L(e^x) = \sum_{n=0}^{\infty} \frac{x^{(n+1)}}{(n+1)!} \int_0^{\infty} e^{-sx} dx = \sum_{n=0}^{\infty} \frac{x^{(n+1)}}{(n+1)!} \cdot \frac{1}{s}$$

Consider $\frac{1}{(1-x)!} = 1 + x + x^2 + x^3 + x^4 + \dots$ for $|x| < 1$

$$\text{Then } \frac{1}{2x!} = \frac{1}{0!} + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \dots$$

$$\text{Then } \frac{1}{1-s} = \frac{1}{s} \cdot \frac{1}{1-\frac{1}{s}} \quad s > 1$$

$$\Rightarrow \frac{1}{s-1} = \sum_{n=0}^{\infty} \frac{1}{s^{n+1}} \quad \text{This implies that}$$

$$L(e^x) = \frac{1}{s-1}$$

$$\text{for } L(\cos x) = \frac{1}{2} \left\{ \frac{1}{s-1} + \frac{1}{s+1} \right\} = \frac{s}{s^2-1}$$

$$\text{This shows that } L(\cos x) = \frac{s}{s^2-1}$$